Derivation of the quantum probability law from minimal non-demolition measurement

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Abstract. One more derivation of the quantum probability rule is presented in order to shed more light on the versatile aspects of this fundamental law. It is shown that the change of state in minimal quantum non-demolition measurement, also known as ideal measurement, implies the probability law in a simple way. Namely, the very requirement of minimal change of state, put in proper mathematical form, gives the well known Lüders formula, which contains the probability rule.

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1. Introduction

The quantum probability law $\mathrm{tr}(E\rho)$ (its so-called trace-rule form) is one of the fundamental pillars of modern physics along with Einstein's famous energy formula $E = mc^2$ and Boltzmann's immortal entropy expression $S = klogW$. Gleason gave a seminal derivation of the quantum probability law in his theorem [1]. Nevertheless, as to transparentness, there is much to be desired. Though the quantum probability law looks simple, there are "wheels within wheels" in it. Therefore, it is important to view it from as many different angles as possible to be able to comprehend the intricacies involved in it.

A number of alternative derivations appeared in the literature. Let me mention just a few.

(i) The approaches based on the so-called eigenvalue-eigenstate link [2], [3], [4];

(ii) The decision-theoretic approaches [5], [6], [7], [8];

(iii) Derivation from operational assumptions [9];

(iv) The approach via entanglement.

The last mentioned approach went under the title "Born's rule from envariance" (environment assisted invariance). There were 4 articles by Zurek [10], [11], [12], [13], who invented the approach, and there were 4 more articles by commentators [14], [15], [16], [17], and finally my own contribution in terms of a complete theory of twin unitaries (the other face of envariance) [18]. The first 8 articles had two restrictions in establishing essentially the trace rule $\mathrm{tr}(E\rho)$ for probability, where E was an event (projector), and ρ was the subsystem density operator: they handled only improper mixtures [19], and did not go beyond the commutation $[E, \rho] = 0$ restriction.

My article emphasized the role of σ-additivity in the derivations from entanglement (the sole assumption in Gleason's theorem). I suggested to surmount the commutation restriction by taking resort to minimal quantum non-demolition (QND) measurement.

Subsequently I have realized that minimal measurement is by itself sufficient to derive the entire trace rule. It has the advantage that it does not require the σ-additivity assumption, and thus it is complementary to Gleason's theorem [1]. This article is devoted to the exposition of the minimal-measurement approach.

The paper is based on the idea that *probabilities are predictions for the statistical weights of definite-result sub-ensembles in measurement.* These are, in the end, detected as relative frequencies.

2. Assumptions of the Derivation

We are dealing with an arbitrary observable A that has a *purely discrete spectrum* $\{a_n : \forall n\}$. We write it in spectral form

$$A = \sum_n a_n P_n, \quad n \neq n' \;\Rightarrow a_n \neq a_{n'}. \tag{1}$$

It will be fixed throughout. We have in mind QND *measurement* of the observable A.

2.1. The assumptions

The *assumptions* of the approach read as follows.

(i) *States are described by density operators* ρ.

By "state" we mean an ensemble of quantum systems prepared by a certain procedure. Any measurement converts the initial state ρ into a final state ρ' (in the so-called non-selective version, when the entire ensemble is considered). The latter is decomposable into states ρ'_n that correspond to the different results a_n of A:

$$\rho' = \sum_n w_n \rho'_n; \quad \forall n: \ \ w_n \geq 0, \ \sum_n w_n = 1. \tag{2}$$

If the measurement is not a QND one, then the states $\{\rho'_n : \forall n, w_n > 0\}$ need not be in any simple relation to A. They correspond to definite pointer positions on the measuring instrument (which we make no use of in this approach). The statistical weights w_n apply both to the states ρ'_n of the selective version (in which definite results are considered), and to the corresponding pointer positions. *By the very definition of measurement, the weights equal the probabilities*:

$$\forall n: \quad w_n = p(a_n, A, \rho) \tag{3}$$

(in obvious notation). In other words, as it was stated in the Introduction, the probabilities $p(a_n, A, \rho)$ are understood to be the predictions for the statistical weights w_n, which become relative frequences when the measurement is performed on the individual systems that make up the ensemble.

QND measurement, by definition, converts an initial state ρ into a final state ρ', which has two properties:

(a) The states ρ'_n that determine the terms in decomposition (2) *are dispersion-free with respect to the observable* A:

$$\forall n, w_n > 0: \quad p(a_n, A, \rho'_n) = 1. \tag{4}$$

(b) If the initial state ρ is itself dispersion-free with respect to A : $\exists n$: $p(a_n, A, \rho) = 1$, then so is the final state, and the sharp value of A is the same: $\rho' = \rho'_n$, but, in general, the initial and the final states need not be equal. (Earlier used synonyms for "non-demolition" were "repeatable", "predictive", "first-kind", etc.)

(ii) *Further, we assume that if and only if a state* ρ *satisfies*

$$\mathrm{tr}(P_n \rho) = 1, \tag{5a}$$

then the probability $p(a_n, A, \rho)$ *of the value* a_n *of the observable* A *in this state is* 1. In other words, we assume the validity of the trace rule for probability-one events.

It is proved in Appendix A that (5a) is (mathematically) *equivalent to*

$$P_n \rho P_n = \rho. \tag{5b}$$

Let us denote by ρ''_n *any state* that has the sharp value a_n of A : $p(a_n, A, \rho''_n) = 1,$ and let us consider the *family of all mixtures*

$$\rho'' \equiv \sum_n v_n \rho''_n; \quad \forall n: \; v_n \geq 0; \; \sum_n v_n = 1. \tag{6}$$

An immediate consequence of (5b) is that decomposition (6) can be rewritten as

$$\rho'' = \sum_n v_n P_n \rho''_n P_n,$$

which, on account of the orthogonality and idempotency of the eigen-projectors $P_n P_{n'} = \delta_{n,n'} P_n,$ implies

$$\rho'' = \sum_n P_n \rho'' P_n. \tag{7}$$

Since (7) is obviously sufficient for (6), also (7) characterizes states that are mixtures of states with definite values of A.

If an initial state ρ and an observable (1) are given, then a *subset* of the family of states (7) are final states of QND measurements.

Our next-to-last *assumption* is:

(iii) *The state* $\bar{\rho}''$ *in the family of states (7) that is* **closest** *to the initial state* ρ *is the final state of a QND measurement of the observable* A. By this, "closest" is meant in the sense of minimal distance, where distance is taken in the Hilbert space $\mathcal{H}_{HS}$ of all Hilbert-Schmidt (HS) operators (cf [20] and Appendix B below). All density operators are HS operators.

In general, also in a proper subset of $\mathcal{H}_{HS},$ in the set of all trace-class operators, for which by definition $\mathrm{tr}\rho < \infty,$ distance is mathematically defined. We take distance in $\mathcal{H}_{HS}$ due to Lemma-C in Appendix C.

Our last assumption is

(iv) *The probabilities* $p(a_n, A, \rho)$ *are the same in all measurements of* A *in* ρ.

2.2. Discussion of the assumptions

Assumptions (i) and (iv) have a basic (almost axiomatic) position in the conceptual structure of quantum mechanics.

Assumption (ii) stipulates the trace law for events that are certain. Here we are on similar grounds as Zurek was [10]-[13], when he set out to derive Born's rule assuming its validity for events that are certain. (In [18] though, when the full power of envariance

was made use of, the trace law under the restriction $[E, \rho] = 0$ was derived with no probability-law assumption to start with.)

Assumption (iii) can be viewed as the definition of minimal (or minimal-disturbance) QND measurement. Namely, "closest" can be understood as "minimally changed".

In the next section we derive $\bar{\rho}''$, and thus we obtain the probabilities.

3. Derivation of the trace rule

We adapt now a former derivation [21] of the Lüders formula [22] to the present purpose.

The argument is very simple. It is based on three almost evident remarks:

Remark 1. The super-operator $\hat{P}_A \equiv \sum_n P_n \ldots P_n$ (cf (1)) is a *projector* in $\mathcal{H}_{HS}$. (The dots show the place where any HS operator $B \in \mathcal{H}_{HS}$ should be in the sum of products when $\hat{P}_A$ is applied to it). One easily shows the claimed Hermiticity and idempotency of $\hat{P}_A$ in $\mathcal{H}_{HS}$ (cf Appendix B).

Let us denote by $\mathcal{S}_A$ the subspace of $\mathcal{H}_{HS}$ onto which $\hat{P}_A$ projects.

Remark 2. As it is obvious from (7), each density operator ρ'' from the family (6) (or (7)) is an element of $\mathcal{S}_A$. And conversely, the family (6) consists of all density operators that are in $\mathcal{S}_A$.

Remark 3. If ρ is a density operator, then so is its projection $\hat{P}_A(\rho)$ (as easily seen).

If ρ is an arbitrary initial state, its closest element in $\mathcal{S}_A$ is its projection into $\mathcal{S}_A$ (cf Appendix D). The projection is a density operator on account of Remark 3. The projection belongs to the family (6) owing to Remark 2. Relation (3) implies that the weights in the projection give the probabilities.

Finally, let us write down the projection.

$$\hat{P}_A(\rho) = \sum_n P_n \rho P_n.$$

This is the well-known formula of Lüders, which gives the change of state in minimal QND measurement (also called ideal measurement) [22].

Making the weights in the preceding relation explicit, one obtains

$$\hat{P}_A(\rho) = \sum_n \Big(\mathrm{tr}(P_n \rho)\Big)\Big(P_n \rho P_n \Big/ [\mathrm{tr}(P_n \rho)]\Big). \tag{8}$$

Relations (3) and (8) give our final result:

$$\forall \rho,\ \forall n : \quad p(a_n, A, \rho) = \mathrm{tr}(P_n \rho). \tag{9}$$

In this way the trace-rule form of the quantum probability law is derived.

Incidentally, if the event is elementary (mathematically, a ray projector) $P_n \equiv\mid \phi\rangle\langle\phi\mid$, then the quantum probability law is known in the form $\langle\phi\mid\rho\mid\phi\rangle$. If also the state is pure (mathematically also a ray projector) $\rho\equiv\mid\psi\rangle\langle\psi\mid$, then one has the transition-probability form $|\langle\phi||\psi\rangle|^2$. (All this obviously follows from the trace rule.)

Appendix A

We prove now the following auxiliary result that sheds light on assumption (ii).

Lemma-A If ρ and P are a density operator and a projector respectively, then $\mathrm{tr}(\rho P)=1$ is *equivalent* to $P\rho P=\rho$.

Proof. It is obvious (by taking the trace) that the latter relation implies the former. Claim of the inverse implication is not quite trivial.

Since every density operator is a trace-class operator, it has a finite or countably infinite discrete positive spectrum $\{r_i : \forall i\}$ (with possible repetitions in the eigenvalues). Hence, it can be written in spectral form as

$$\rho=\sum_i r_i\mid i\rangle\langle i\mid, \qquad (A.1)$$

where $\mid i\rangle$ is an eigenvector corresponding to the eigenvalue r_i.

The relation $\mathrm{tr}(\rho P)=1$ implies $\mathrm{tr}(\rho P^\perp)=0$ ($P^\perp\equiv 1-P$). Substituting (A.1) in the latter relation, one obtains $\sum_i r_i\langle i\mid P^\perp\mid i\rangle=0$. On account of the positivity $\forall i:\ r_i>0$, and the easily seen non-negativity $\forall i:\ \langle i\mid P^\perp\mid i\rangle\geq 0$, one further has $\forall i:\ 0=\langle i\mid P^\perp\mid i\rangle=||P^\perp\mid i\rangle||^2$, as well as $\forall i:\ P^\perp\mid i\rangle=0$, and $\forall i:\ P\mid i\rangle=\mid i\rangle$. Then, applying $P\dots P$ to (A.1), one obtains the second relation in Lemma-A. □

Appendix B

By definition, linear operators A in a complex separable Hilbert space are Hilbert-Schmidt ones if $\mathrm{tr}(A^\dagger A)<\infty$ ($A^\dagger$ being the adjoint of A). The scalar product in the Hilbert space $\mathcal{H}_{HS}$ of all linear Hilbert-Schmidt operators is $\Big(A,B\Big)\equiv\mathrm{tr}(A^\dagger B)$ (cf the Definition after Theorem VI.21 and problem VI.48(a) in [20]).

Appendix C

Let $\mathcal{H}$ be a separable, complex Hilbert space, and $\mathcal{H}_{HS}$ the Hilbert space of all linear Hilbert-Schmidt operators in it (cf Appendix B). Let, further, $\mid\psi\rangle$, and $\mid\phi\rangle$ be two arbitrary unit vectors in $\mathcal{H}$. The square of the *distance* between them in $\mathcal{H}$ is

$$\Big[d_{\mathcal{H}}\Big(\mid\psi\rangle,\mid\phi\rangle\Big)\Big]^2\equiv||\mid\psi\rangle-\mid\phi\rangle||^2=\Big(\langle\psi\mid-\langle\phi\mid\Big)\Big(\mid\psi\rangle-\mid\phi\rangle\Big)=2-2Re\Big(\langle\phi||\psi\rangle\Big). \quad (C.1)$$

It depends on the relative phase between the two vectors.

Definition-C (i) We make the convention that, whenever the distance between two unit vectors in $\mathcal{H}$ is in question, it is understood that the relative phase is chosen so that the distance in (C.1) is minimal, i. e., that

$$\langle\phi||\psi\rangle \geq 0. \tag{C.2}$$

(ii) We use the word "closer" in the sense of "not farther", i. e., as $\leq$, and not as $<$.

Lemma-C Let $\mid\psi\rangle$, $\mid\phi\rangle$, and $\mid\chi\rangle$ be three arbitrary unit vectors in $\mathcal{H}$. Then, taking the phase factors of $|\phi\rangle$ and $|\chi\rangle$ in accordance with Definition-C (i), the former is closer than the latter to the state vector $|\psi\rangle$ in $\mathcal{H}$, *if and only if* the corresponding pure state $\mid\phi\rangle\langle\phi\mid$ is closer than $\mid\chi\rangle\langle\chi\mid$ to $\mid\psi\rangle\langle\psi\mid$ in $\mathcal{H}_{HS}$. In other words, closer in $\mathcal{H}$ (observing Definition-C (i)) is the case if and only if it is true for the corresponding ray projectors in $\mathcal{H}_{HS}$.

Proof. In view of (C.1) and Definition-C (i), $|\phi\rangle$ is closer to $|\psi\rangle$ than $|\chi\rangle$ is to $|\psi\rangle$ if and only if

$$\Big(2-2|\langle\phi||\psi\rangle|\Big) \leq \Big(2-2|\langle\chi||\psi\rangle|\Big) \;\;\Leftrightarrow\;\; |\langle\phi||\psi\rangle| \geq |\langle\chi||\psi\rangle|. \tag{C.3}$$

On the other hand, one has

$$\Big[d_{\mathcal{H}_{HS}}\Big(\mid\psi\rangle\langle\psi\mid, \mid\phi\rangle\langle\phi\mid\Big)\Big]^2 = \mathrm{tr}\Big[\Big(\mid\psi\rangle\langle\psi\mid - \mid\phi\rangle\langle\phi\mid\Big)^2\Big] = 2-2|\langle\phi||\psi\rangle|^2. \tag{C.4}$$

Hence, the pure state $|\phi\rangle\langle\phi|$ is "closer" to $|\psi\rangle\langle\psi|$ than $|\chi\rangle\langle\chi|$ is to $|\psi\rangle\langle\psi|$ in $\mathcal{H}_{HS}$ if and only if

$$|\langle\phi||\psi\rangle|^2 \geq |\langle\chi||\psi\rangle|^2.$$

Finally, since an inequality between two non-negative numbers holds true if and only if the same inequality is valid between their squares, one can see from (C.3) and (C.4) that Lemma-C is proved. $\Box$

Appendix D

Now we prove (for completeness) a very elementary auxiliary lemma.

Lemma-D Let $\mathcal{H}$ and $\mathcal{S}$ be a separable (finite or infinite dimensional) complex Hilbert space and a subspace in it respectively. Let, further, P be the projector onto $\mathcal{S}$. For every element $a \in \mathcal{H}$, there is a unique element $\bar{b} \in \mathcal{S}$ that is closest to a among all elements $b \in \mathcal{H}$. It is $\bar{b} \equiv Pa$. By this, "closest" is meant in the sense of minimal distance $||a-b||$.

Proof. For every $a \in \mathcal{H}$, and every $b \in \mathcal{S}$, one can utilize the orthogonality between the vectors from the orthocomplement of $\mathcal{S}$ and those from $\mathcal{S}$ itself:

$$||a-b||^2 = ||(a-Pa)+(Pa-b)||^2 = ||a-Pa||^2 + ||Pa-b||^2.$$

This is minimal with respect to the choice of $b \in \mathcal{S}$ if and only if $b \equiv Pa$ because whenever $b \in \mathcal{S}$, $b \neq Pa$, $||Pa-b||^2 > 0$. $\Box$

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